

DERIVED AND ABELIAN EQUIVALENCE OF K3 SURFACES

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Any isomorphism $X \cong X'$ between two schemes induces an equivalence $\mathrm{Coh}(X) \cong \mathrm{Coh}(X')$ between their abelian categories of coherent sheaves. Due to a classical result of Gabriel [5] the converse holds true as well. Thus,

$$X \cong X' \iff \mathrm{Coh}(X) \cong \mathrm{Coh}(X').$$

Is there a similar statement when isomorphism of schemes is replaced by derived equivalence? More precisely, can one naturally associate abelian categories to two varieties X and X' such that the varieties are derived equivalent if and only if there exists an equivalence between the abelian categories?

We will restrict to complex K3 surfaces and prove

Theorem 0.1. *Two complex projective K3 surfaces X and X' are derived equivalent if and only if there exist complexified Kähler classes $B + i\omega$ and $B' + i\omega'$ on X respectively X' such that the two abelian categories $\mathcal{A}_X(\exp(B + i\omega))$ and $\mathcal{A}_{X'}(\exp(B' + i\omega'))$ are equivalent. Thus,*

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(X') \iff \mathcal{A}_X(\exp(B + i\omega)) \cong \mathcal{A}_{X'}(\exp(B' + i\omega')).$$

Here, $\mathrm{D}^b(X)$ is the bounded derived category $\mathrm{D}^b(\mathrm{Coh}(X))$ and the equivalence on the left hand side is linear and exact. By definition $\mathcal{A}(\exp(B + i\omega))$ is the full subcategory of all complexes $F^\bullet \in \mathrm{D}^b(X)$ with cohomology concentrated in degree -1 and zero and such that $\mathcal{H}^{-1}(F^\bullet)$ is torsion free with $\mu_{\max} \leq (B.\omega)$ and the torsion free part of $\mathcal{H}^0(F^\bullet)$ satisfies $\mu_{\min} > (B.\omega)$. (For the notation and the details of this definition see Section 1.) In our situation, $B + i\omega$ can be taken as a complexified ample class, i.e. $B + i\omega \in \mathrm{Pic}(X) \otimes \mathbb{C}$ with $\omega \in \mathrm{Pic}(X)$ ample. The two directions of the theorem are proved in Section 1 (Cor. 1.3) respectively Section 5 (Cor. 5.3).

The abelian category $\mathcal{A}_X(\exp(B + i\omega))$ is the heart of a t -structure on $\mathrm{D}^b(X)$ that has been studied by Bridgeland in [4] in his quest for stability conditions on the triangulated category $\mathrm{D}^b(X)$. Bridgeland introduced the concept of stability conditions, in [1] in an effort to understand Douglas's work on stability of branes. Roughly, a stability condition on a triangulated category consists of a t -structure and a stability function on its heart satisfying the Harder–Narasimhan property. It is surprisingly difficult to construct stability conditions on the derived category of a higher dimensional projective Calabi–Yau variety. In [4] Bridgeland considers the case

of K3 surfaces. In order to get started he needs to construct explicit examples of stability conditions and the heart of those are the abelian categories $\mathcal{A}(\exp(B + i\omega))$. (The abelian category $\text{Coh}(X)$ never occurs as the heart of a stability condition.)

This paper grew out of the attempt to understand the geometric meaning of the abelian categories $\mathcal{A}(\exp(B + i\omega))$.

The naive idea to prove Gabriel's result that any equivalence $\text{Coh}(X) \cong \text{Coh}(X')$ induces an isomorphism of the K3 surfaces X and X' is the following. First note that the *simple* objects (or the *minimal* objects, as we will call them, see Section 1) of $\text{Coh}(X)$ are the structure sheaves $k(x)$ of closed points $x \in X$. Since this is an intrinsic notion, any equivalence of abelian categories sends minimal objects to minimal objects. Thus, the equivalence $\text{Coh}(X) \cong \text{Coh}(X')$ induces a bijection $X \cong X'$ and, in order to fully prove Gabriel's result, one only has to show that this bijection is a morphism.

From this point of view it is natural to wonder how the minimal objects of $\mathcal{A}(\exp(B + i\omega))$ look like. This question is interwoven with Theorem 0.1 and we shall give the following complete classification in Section 2 (Prop. 2.2):

Theorem 0.2. *For a K3 surface X the minimal objects in $\mathcal{A}(\exp(B + i\omega))$ are precisely the objects*

- $k(x)$, where $x \in X$ is a closed point and
- $F[1]$, where F is a μ -stable locally free sheaf with $\mu(F) = (B.\omega)$.

Thus, any equivalence between $\mathcal{A}_{X'}$ and \mathcal{A}_X will either induce an isomorphism $X' \cong X$ or will map closed points in X' to shifted μ -stable vector bundles on X . In order to combine both theorems, we have to prove a stronger version of Orlov's well-known result saying that two K3 surfaces are derived equivalent if and only if one is a moduli space of stable sheaves on the other. In Proposition 4.1 we actually prove that in Orlov's result one can replace 'stable' by ' μ -stable' and 'sheaves' by 'vector bundles'.

The abelian category $\mathcal{A}(\exp(B + i\omega))$ plays a decisive role in D-equivalence of K3 surfaces, but it also appears naturally from a differential-geometric point of view. The minimal objects of $\mathcal{A}(\exp(i\omega))$, besides the point sheaves, are (shifted) hyperholomorphic bundles, i.e. bundles that are holomorphic with respect to all hyperkähler rotations (with respect to ω) of the original complex structure. A short discussion of this point of view is included in Section 6.

The last section of this paper proves stability of Fourier–Mukai transforms of certain μ -stable vector bundles. The main result not only yields stability in cases not covered by existing result, but it gives, maybe more interestingly, a conceptual explanation when and why stability of a Fourier–Mukai transform of a μ -stable vector bundle can be expected with Mukai vector $v = (r, \ell, s)$. There we prove

Theorem 0.3. *There exists a polarization H' on X' such that for any μ -stable vector bundle E on X with $\mu(E) = -(\ell.H)/r$ one has either*

- $\Phi(E) \cong k(y)[-2]$ if $[E^\vee] \in M_H(v)$ or otherwise
- $\Phi(E) \cong F[-1]$ with F a $\mu_{H'}$ -stable vector bundle on X' .

Acknowledgements. I am grateful to Tom Bridgeland for writing [4], answering my questions on it, and reminding me of some of the many useful results in Mukai's paper [14]. Discussions with Alastair King were helpful to get my ideas straight.

1. ABELIAN EQUIVALENCE YIELDS DERIVED EQUIVALENCE

A *torsion pair* in an abelian category \mathcal{C} is a pair of full subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$ and such that every object $E \in \mathcal{C}$ fits into a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Following [6] one associates to a given torsion pair $(\mathcal{T}, \mathcal{F})$ in \mathcal{C} a *t-structure* on the bounded derived category $\text{D}^b(\mathcal{C})$ by setting

$$\text{D}^{\leq 0} := \{F^\bullet \in \text{D}^b(\mathcal{C}) \mid H^i(F^\bullet) = 0, i > 0; H^0(F^\bullet) \in \mathcal{T}\}$$

and

$$\text{D}^{\geq 0} := \{F^\bullet \in \text{D}^b(\mathcal{C}) \mid H^i(F^\bullet) = 0, i < -1; H^{-1}(F^\bullet) \in \mathcal{F}\}.$$

Its heart, also called the *tilt* of \mathcal{C} , is the abelian category

$$\begin{aligned} \mathcal{A}(\mathcal{T}, \mathcal{F}) &:= \text{D}^{\leq 0} \cap \text{D}^{\geq 0} \\ &= \{F^\bullet \in \text{D}^b(\mathcal{C}) \mid H^i(F^\bullet) = 0, i \neq 0, -1; H^0(F^\bullet) \in \mathcal{T}; H^{-1}(F^\bullet) \in \mathcal{F}\}. \end{aligned}$$

Thus, any object in $\mathcal{A}(\mathcal{T}, \mathcal{F})$ is isomorphic to a complex of the form

$$F^{-1} \xrightarrow{\varphi} F^0$$

with $\text{coker}(\varphi) \in \mathcal{T}$ and $\text{ker}(\varphi) \in \mathcal{F}$. Note that in particular $\mathcal{F}[1]$ and \mathcal{T} are both naturally contained in $\mathcal{A}(\mathcal{T}, \mathcal{F})$. Moreover, $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{A}(\mathcal{T}, \mathcal{F})$ whose tilt is $\mathcal{C}[1]$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *tilting* if every object in \mathcal{C} is a subobject of an object in \mathcal{T} . Similarly, $(\mathcal{T}, \mathcal{F})$ is *cotilting* if every object in \mathcal{C} is a quotient of an object in \mathcal{F} . In the latter case, every object in \mathcal{C} admits a resolution of length two by objects in \mathcal{F} . Indeed, any subobject of an object in \mathcal{F} is in \mathcal{F} .

Suppose $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair. Then the natural inclusion $\mathcal{F} \subset \mathcal{C}$ induces an exact equivalence $\text{D}^b(\mathcal{F}) \xrightarrow{\sim} \text{D}^b(\mathcal{C})$ (see [2, Lemma 5.4.2]). Similar, if $(\mathcal{T}, \mathcal{F})$ is tilting, then $\text{D}^b(\mathcal{T}) \xrightarrow{\sim} \text{D}^b(\mathcal{C})$ is an equivalence. Using that $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in \mathcal{C} if and only if $(\mathcal{F}[1], \mathcal{T})$ is a tilting pair in $\mathcal{A}(\mathcal{T}, \mathcal{F})$ (see [6, Prop. I.3.2]) one obtains for a cotilting pair $(\mathcal{T}, \mathcal{F})$ two exact equivalences

$$\text{D}^b(\mathcal{F}) \cong \text{D}^b(\mathcal{C}) \text{ and } \text{D}^b(\mathcal{F}) \cong \text{D}^b(\mathcal{F}[1]) \cong \text{D}^b(\mathcal{A}(\mathcal{T}, \mathcal{F})).$$

This yields

Proposition 1.1. *For any cotilting torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category \mathcal{C} there exists an exact equivalence*

$$\mathrm{D}^b(\mathcal{C}) \cong \mathrm{D}^b(\mathcal{A}(\mathcal{T}, \mathcal{F})).$$

The result was first proved in [6] under additional assumptions (e.g. the existence of enough injectives in \mathcal{C}) and in the above form in [2, Prop. 5.4.3].

Let us now turn to the more concrete situation where the abelian category \mathcal{C} is the category $\mathrm{Coh}(X)$ of coherent sheaves on a smooth projective variety X of dimension n .

We fix a polarization H or, more generally, a Kähler class ω , so that degree $\deg(F)$ and slope $\mu(F)$ of a coherent sheaf F on X can be defined as $\deg(F) := \int_X c_1(F) \cdot H^{n-1}$ (respectively $\int_X c_1(F) \cdot \omega^{n-1}$) and $\mu(F) := \deg(F)/\mathrm{rk}(F)$.

The *Harder–Narasimhan-filtration* (HN-filtration for short) of a coherent sheaf F is the unique filtration

$$0 \subset F_0 \subset F_1 \subset \dots \subset F_n = F$$

that satisfies the following conditions: i) F_0 is the torsion part of F , ii) The quotients F_{i+1}/F_i are torsion free and μ -semistable for $i = 0, \dots, n-1$, and iii) $\mu(F_1/F_0) > \dots > \mu(F_n/F_{n-1})$.

Existence and uniqueness are easy to prove, see e.g. [8, Thm. 1.6.7]. One denotes $\mu_{\max}(F) := \mu(F_1/F_0)$ and $\mu_{\min}(F) := \mu(F_n/F_{n-1})$ provided F is not torsion.

For $\beta \in \mathbb{R}$ one introduces the full subcategories

$$\mathcal{T}(\beta), \mathcal{F}(\beta) \subset \mathrm{Coh}(X).$$

By definition $\mathcal{T}(\beta)$ is the category of all coherent sheaves F with $\mu_{\min}(F) > \beta$ or F is a torsion sheaf and $\mathcal{F}(\beta)$ is the category of all torsion free coherent sheaves F with $\mu_{\max}(F) \leq \beta$ or $F \cong 0$.

The following is an immediate consequence of the basic properties of μ -semistable sheaves and the existence of the HN-filtration.

Proposition 1.2. *With the above notation $\mathcal{T}(\beta), \mathcal{F}(\beta) \subset \mathrm{Coh}(X)$ is a torsion pair.* \square

Torsion pairs of this form have been introduced by A. Schofield and were later studied for curves in [18] and for K3 surfaces in [4].

Let us denote the heart of the induced t -structure by $\mathcal{A}(\beta)$ (or, $\mathcal{A}_X(\beta)$ if the dependence on X needs to be stressed). Thus, $\mathcal{A}(\beta)$ is the following full subcategory of $\mathrm{D}^b(X) = \mathrm{D}^b(\mathrm{Coh}(X))$:

$$\mathcal{A}(\beta) := \{ F^{-1} \xrightarrow{\varphi} F^0 \mid \ker(\varphi) \in \mathcal{F}(\beta), \mathrm{coker}(\varphi) \in \mathcal{T}(\beta) \}.$$

Note that, although not reflected by the notation, $\mathcal{A}(\beta)$ depends on β and on the chosen polarization (respectively Kähler class).

Corollary 1.3. *Suppose X and X' are smooth projective varieties endowed with polarizations H respectively H' (or Kähler classes ω respectively ω'). If for two real numbers β, β' the abelian categories $\mathcal{A}_X(\beta)$ and $\mathcal{A}_{X'}(\beta')$ are equivalent, then X and X' are derived equivalent. In other words,*

$$\mathcal{A}_X(\beta) \cong \mathcal{A}_{X'}(\beta') \implies \mathrm{D}^b(X) \cong \mathrm{D}^b(X').$$

Remark 1.4. HN-filtrations not only exist with respect to μ -semistability (see [8, Ch. 1]). As only the formal properties were used, the above discussion goes through unchanged for e.g. Gieseker stability for which the slope $\mu(F)$ is replaced by the Hilbert polynomial $\chi(F(n))$.

In the following we shall be interested in the case of an algebraic K3 surface X . The K3 surface X will be endowed with a Kähler class $\omega \in \mathrm{NS}(X) \otimes \mathbb{R} \cong H^{1,1}(X, \mathbb{Z}) \otimes \mathbb{R}$ and a B-field $B \in \mathrm{NS}(X) \otimes \mathbb{R}$. Then let $\beta := (B, \omega)$.

Instead of considering B and ω or the complexified Kähler class $B + i\omega$, one more naturally uses $\exp(B + i\omega) = 1 + (B + i\omega) + (B + i\omega)^2/2 \in H^*(X, \mathbb{C})$, which can be seen as a generalized Calabi–Yau structure on X . This point of view fits nicely with the picture proposed by mirror symmetry. For a discussion see [10].

Changing the notation of [4] slightly we shall thus write

$$\mathcal{T} := \mathcal{T}(\exp(B + i\omega)) := \mathcal{T}(\beta), \quad \mathcal{F} := \mathcal{F}(\exp(B + i\omega)) := \mathcal{F}(\beta)$$

and

$$\mathcal{A}(\exp(B + i\omega)) := \mathcal{A}(\beta).$$

In particular Corollary 1.3 for two K3 surfaces X and X' reads:

$$\mathcal{A}_X(\exp(B + i\omega)) \cong \mathcal{A}_{X'}(\exp(B' + i\omega')) \implies \mathrm{D}^b(X) \cong \mathrm{D}^b(X').$$

Using the Mukai vector $v(F) = \mathrm{ch}(F)\sqrt{\mathrm{td}(X)} = (r, \ell, s)$, Bridgeland introduces

$$Z(F) := \langle v(F), \exp(B + i\omega) \rangle$$

in order to construct a stability function on the abelian category $\mathcal{A}(\exp(B + i\omega))$ (see [4, Sect. 5]). Here, $\langle \cdot, \cdot \rangle$ is the Mukai pairing.

Clearly,

$$\mathrm{Im} \langle v(F), \exp(B + i\omega) \rangle = (\ell, \omega) - r(B, \omega) = (\ell, \omega) - r\beta.$$

Thus, for $r \neq 0$, the slope $Z(F)$ is contained in the upper half plane if and only if $\mu(F) > \beta$. Also, for $F \in \mathcal{T}$ one has $\mathrm{Im}(Z(F)) \geq 0$ and, similarly, if $F \in \mathcal{F}$, then $\mathrm{Im}(Z(F)) \leq 0$. Using $v(F[1]) = -v(F)$, this shows that $\mathrm{Im}(Z(F^\bullet)) \geq 0$ for all $F^\bullet \in \mathcal{A}(\exp(B + i\omega))$.

Remark 1.5. In [4, Lemma 5.2] it is shown that $Z(F^\bullet) \in \mathbb{R}_{>0} \exp(i\pi\phi(F^\bullet))$ with the phase $\phi(F^\bullet)$ satisfying $0 < \phi(F^\bullet) \leq 1$ holds for all $0 \neq F^\bullet \in \mathcal{A}(\exp(B + i\omega))$ if and only if $Z(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves F . Note that the latter holds as soon as $(\omega, \omega) > 2$.

For later use we note that $\mathrm{Im}(Z(k(x))) = 0$ for any closed point $x \in X$ and $\mathrm{Im}(Z(F[1])) = 0$ for any μ -stable vector bundle F with $\mu(F) = \beta$. Under the assumption of the remark, this is equivalent to $\phi(k(x)) = \phi(F[1]) = 1$.

2. MINIMAL OBJECTS IN \mathcal{A}

The aim of this section is to classify minimal objects in $\mathcal{A}(\exp(B + i\omega))$ (modulo a technical result postponed to the next section).

Recall that a non-trivial object A in an abelian category \mathcal{A} is called *minimal* if any surjection $A \twoheadrightarrow B$ with $B \neq 0$ is an isomorphism. Equivalently, A is minimal if and only if every injection $0 \neq C \hookrightarrow A$ is an isomorphism, i.e. A has no proper subobjects. Usually, objects of this type are called *simple*, but ‘simple’ for a sheaf $F \in \mathrm{Coh}(X)$ has also a different meaning, i.e. that $\mathrm{End}(F) = k$, so we rather use ‘minimal’ instead.

Here are a few easy observations. Suppose A is minimal and $\varphi : A \rightarrow B$ is a morphism. Then either $\varphi = 0$ or φ is injective. If in addition B is minimal as well, then either $\varphi = 0$ or φ is an isomorphism.

Example 2.1. As was mentioned earlier in the introduction, the minimal objects in $\mathrm{Coh}(X)$ are the point sheaves $k(x)$ with $x \in X$ a closed point. These minimal objects have the additional property that any non-trivial sheaf $F \in \mathrm{Coh}(X)$ admits a surjection $F \twoheadrightarrow k(x)$ for some $x \in X$.

Note however that they do not generate $\mathrm{Coh}(X)$. Recall that a collection of objects in an abelian category generates the category if every object admits a filtration whose quotients are isomorphic to objects in the collection.

We shall be interested in the abelian category $\mathcal{A} := \mathcal{A}(\exp(B + i\omega))$ on a K3 surface X , which is by definition a full subcategory of the derived category $\mathrm{D}^b(X)$ obtained as a tilt of $\mathrm{Coh}(X)$ with respect to a torsion pair $(\mathcal{T}, \mathcal{F})$.

As this will be frequently used in the following discussion, we recall the following standard fact (see [13, p. 415]): Let \mathcal{A} be the heart of a t -structure on a triangulated category D . If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in \mathcal{A} , then there exists a map $C \rightarrow A[1]$ such that $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle in D . Conversely, if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle in D with objects A, B, C in \mathcal{A} , then $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in \mathcal{A} .

In the following $B + i\omega \in \mathrm{NS}(X)_{\mathbb{C}}$ is a complexified Kähler class, i.e. $\omega \in \mathrm{NS}(X)_{\mathbb{R}}$ is a Kähler class and $B \in \mathrm{NS}(X)_{\mathbb{R}}$ is arbitrary. Stability is considered with respect to ω and we do not assume ω or B to be rational.

Proposition 2.2. *The minimal objects in $\mathcal{A}(\exp(B + i\omega))$ are precisely the objects*

- $k(x)$, where $x \in X$ is a closed point and
- $F[1]$, where F is a μ -stable locally free sheaf with $\mu(F) = (B, \omega)$.

Proof. To shorten the notation we write $(\mathcal{T}, \mathcal{F})$ for the torsion pair induced by $\exp(B + i\omega)$. Similarly, $\mathcal{A} := \mathcal{A}(\exp(B + i\omega))$. As before, we use $\beta := (B, \omega)$.

i) Point sheaves are minimal. First, $k(x) \in \mathcal{T} \subset \mathcal{A}$ for any closed point $x \in X$. Suppose $k(x) \twoheadrightarrow F^\bullet$ is a non-trivial surjection in \mathcal{A} which we complete to a short exact sequence

$$0 \longrightarrow E^\bullet \longrightarrow k(x) \longrightarrow F^\bullet \longrightarrow 0$$

in \mathcal{A} . Considered as a distinguished triangle in $D^b(X)$ it yields the long exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(E^\bullet) \longrightarrow 0 \longrightarrow \mathcal{H}^{-1}(F^\bullet) \longrightarrow \mathcal{H}^0(E^\bullet) \longrightarrow k(x) \longrightarrow \mathcal{H}^0(F^\bullet) \longrightarrow 0.$$

Thus, $E^\bullet \cong E$ with $E \in \mathcal{T}$. Moreover, $\mathcal{H}^{-1}(F^\bullet)$ and $\mathcal{H}^0(E^\bullet) \cong E$ are isomorphic on $X \setminus \{x\}$. Since $\mathcal{H}^{-1}(F^\bullet) \in \mathcal{F}$, this yields for $\mathcal{H}^{-1}(F^\bullet) \neq 0$ the contradiction

$$\beta < \mu_{\min}(E) = \mu_{\min}(\mathcal{H}^{-1}(F^\bullet)) \leq \mu_{\max}(\mathcal{H}^{-1}(F^\bullet)) \leq \beta.$$

Hence, $\mathcal{H}^{-1}(F^\bullet) = 0$, i.e. $F^\bullet \cong \mathcal{H}^0(F^\bullet) \neq 0$. By minimality of $k(x)$ as an object in $\text{Coh}(X)$, the surjection $k(x) \twoheadrightarrow \mathcal{H}^0(F^\bullet)$ is an isomorphism and hence the surjection $k(x) \twoheadrightarrow F^\bullet$ in \mathcal{A} is one.

ii) Stable vector bundles of slope β are minimal. Let F be a μ -stable locally free sheaf with $\mu(F) = \beta$. Then $F[1] \in \mathcal{A}$ by definition of \mathcal{A} . Consider a short exact sequence

$$(1) \quad 0 \longrightarrow G^\bullet \longrightarrow F^\bullet[1] \longrightarrow E^\bullet \longrightarrow 0$$

in \mathcal{A} . In order to show that $F[1] \in \mathcal{A}$ is minimal, one proves that either $G^\bullet = 0$ or $E^\bullet = 0$. The long exact cohomology sequence of (1) considered as a distinguished triangle in $D^b(X)$ reads

$$0 \longrightarrow \mathcal{H}^{-1}(G^\bullet) \longrightarrow F \xrightarrow{\varphi} \mathcal{H}^{-1}(E^\bullet) \longrightarrow \mathcal{H}^0(G^\bullet) \longrightarrow 0 \longrightarrow \mathcal{H}^0(E^\bullet) \longrightarrow 0.$$

Hence, $E^\bullet \cong E[1]$, where E is torsion free with $\mu_{\max}(E) \leq \beta$. Consider the morphism $\varphi : F \longrightarrow \mathcal{H}^{-1}(E^\bullet) \cong E$ and its image E' . If φ is neither trivial nor injective, then μ -stability of F yields the contradiction $\beta = \mu(F) < \mu(E') \leq \mu_{\max}(E) \leq \beta$.

If $\varphi = 0$, then $\mathcal{H}^{-1}(E^\bullet) \cong \mathcal{H}^0(G^\bullet)$ and $(\mathcal{H}^{-1}(G^\bullet) \cong F)$. Since the only common object of \mathcal{T} and \mathcal{F} is the trivial sheaf, the latter is only possible if $E \cong 0$. Hence, $E^\bullet \cong 0$.

If φ is injective, then $\mathcal{H}^{-1}(G^\bullet) = 0$, i.e. $G^\bullet \cong G := \mathcal{H}^0(G^\bullet)$ and we get a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

in $\text{Coh}(X)$ with $E \in \mathcal{F}$ and $G \in \mathcal{T}$. As $\mu_{\max}(E) \leq \beta$, the sheaf G must be torsion. If G is not concentrated in dimension zero, then $\deg(G) > 0$ and hence $\mu(E) = (\deg(G) + \deg(F))/\text{rk}(F) > \mu(F) = \beta$ contradicting

$\mu_{\max}(E) \leq \beta$. If G is concentrated in dimension zero, then $\text{Ext}^1(G, F) \cong \text{Ext}^1(F, G)^* \cong H^1(X, F^\vee \otimes G)^* = 0$, for F is locally free. Thus, $E \cong F \oplus G$, which for $G \neq 0$ contradicts the torsion freeness of E .

iii) That's all. Suppose $F^\bullet \in \mathcal{A}$ is minimal and $F^\bullet \not\cong k(x)$ for all closed points $x \in X$. The first two of the following claims, which we include here for completeness sake, correspond to b) of Lemma 6.1 in [4].

Claim 1. $\mathcal{H}^0(F^\bullet) = 0$.

Otherwise, there exists a surjection $\mathcal{H}^0(F^\bullet) \twoheadrightarrow k(x)$ in $\text{Coh}(X)$ and hence a non-trivial morphism $F^\bullet \rightarrow k(x)$ in \mathcal{A} . As both objects are minimal, it would necessarily be an isomorphism.

Hence, $F^\bullet \cong F[1]$ with $F \in \mathcal{F}$.

Claim 2. F is locally free.

If not, then there exists a short exact sequence

$$0 \longrightarrow F \longrightarrow F' \longrightarrow k(x) \longrightarrow 0$$

in $\text{Coh}(X)$ with F' still torsion free. Hence, $F' \in \mathcal{F}$. Thus, the induced distinguished triangle

$$k(x) \longrightarrow F[1] \longrightarrow F'[1]$$

yields a short exact sequence in \mathcal{A} , contradicting the minimality of $F[1]$.

Claim 3. F is μ -stable.

If not, then there exists a short exact sequence

$$(2) \quad 0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with F_1, F_2 torsion free, non-trivial and such that $\mu(F_1) \geq \mu(F) \geq \mu(F_2)$. Since $F \in \mathcal{F}$, also $F_1, F_2 \in \mathcal{F}$. Therefore the shift of (2) yields a short exact sequence in \mathcal{A} contradicting the minimality of F^\bullet .

Claim 4. $\mu(F) = \beta$.

Here we use Proposition 3.1 which shall be proved in the next section. It asserts that as soon as $\mu(F) < \beta$, there exists a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

of μ -stable vector bundles with $\mu(E) \leq \beta < \mu(G)$. The induced distinguished triangle

$$G \longrightarrow F[1] \longrightarrow E[1]$$

is then a short exact sequence in \mathcal{A} which again contradicts the minimality of $F[1]$. For an argument that does not make use of the full Proposition 3.1, whose proof is unpleasantly long, see Remark 3.3. \square

In the ‘irrational’ situation, the category $\mathcal{A}(\exp(B + i\omega))$ has the same minimal objects as $\text{Coh}(X)$. More precisely, the proposition yields:

Corollary 2.3. *Suppose the complex polarization $B + i\omega$ is chosen such that $\omega \in \text{NS}(X)_\mathbb{Q}$ and $(B, \omega) \notin \mathbb{Q}$. Then the only minimal objects in $\mathcal{A}(\exp(B + i\omega))$ are the point sheaves $k(x)$.*

Proof. If ω is rational, then for any sheaf F the slope $\mu(F) \in \mathbb{Q}$. In particular, there are no μ -stable vector bundles with $\mu(F) = (B.\omega)$. \square

Remark 2.4. In the rational case, i.e. $B, \omega \in \text{NS}(X)_{\mathbb{Q}}$, the minimal objects described by the proposition share the property of the minimal objects in $\text{Coh}(X)$ alluded to in Example 2.1: Every non-trivial object in $\mathcal{A}(\exp(B + i\omega))$ admits a surjection onto a minimal object.

Suppose $F^{\bullet} \in \mathcal{A}$ with $\mathcal{H}^0(F^{\bullet}) \neq 0$. Then, any surjection $\mathcal{H}^0(F^{\bullet}) \twoheadrightarrow k(x)$ induces a surjection $F^{\bullet} \twoheadrightarrow k(x)$ in \mathcal{A} . If $\mathcal{H}^0(F^{\bullet}) = 0$, then $F^{\bullet} \cong F[1]$. We may assume that F is μ -stable and locally free. If $\mu(F) = (B.\omega)$, then $F[1] \in \mathcal{A}$ is minimal. If not, one uses Remark 3.4, iii), which says that there always exists a short exact sequence $0 \rightarrow G \rightarrow F[1] \rightarrow E[1] \rightarrow 0$ in \mathcal{A} with E μ -stable, locally free and such that $\mu(E) = \beta$.

3. STABLE EXTENSIONS: A TECHNICAL FACT

Let us fix a Kähler class ω on a projective K3 surface X and consider degree \deg and slope μ with respect to ω .

Proposition 3.1. *Fix $\beta \in \mathbb{R}$. If F is a μ -stable vector bundle on X with*

$$(3) \quad \mu(F) < \beta,$$

then there exists a short exact sequence

$$(4) \quad 0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

of μ -stable vector bundles with

$$(5) \quad \mu(E) \leq \beta < \mu(G).$$

The inequalities in (5) impose numerical conditions on the line bundles obtained as the determinants of the bundles in (4). The existence of the line bundles is shown first.

In the following we let $L := \det(F)$ and $r := \text{rk}(F)$.

Lemma 3.2. *Suppose $\mu(F) < \beta$. Then there exist a line bundle L' and an integer $r' > 0$ such that*

$$(6) \quad \frac{\deg(L) + \deg(L')}{r + r'} \leq \beta < \frac{\deg(L')}{r'}.$$

Moreover, L' can be chosen such that L and L' are linearly dependent.

Proof. Suppose we can prove the existence of L' and r' satisfying (6) for a twist $F \otimes H$ of F by some (ample) line bundle H . Then $L' \otimes H^{-r'}$ would satisfy (6) for F itself. Note that $\mu(F) < \beta$ if and only if $\mu(F \otimes H) < \beta + \deg(H)$. Therefore, we may add the simplifying assumption $0 < \deg(L)$.

The line bundle L' will be chosen of the form $L' := L^{\ell'}$ for some integer $\ell' > 0$. Dividing (3) and (6) by $\deg(L)$ the two inequalities become (with $\tilde{\beta} := \beta / \deg(L)$):

$$1 < r\tilde{\beta} \text{ and } \frac{1 + \ell'}{r + r'} \leq \tilde{\beta} < \frac{\ell'}{r'}.$$

The latter is equivalent to

$$\tilde{\beta} < \frac{\ell'}{r'} \leq \tilde{\beta} + \frac{\tilde{\beta}r - 1}{r'}.$$

To conclude recall the standard fact that for any $x \in \mathbb{R}_{>0}$ and all $\varepsilon > 0$ there exists a rational number $\frac{a}{b}$ with a, b positive integers such that $0 < \frac{a}{b} - x < \frac{\varepsilon}{b}$. Apply this to $x := \tilde{\beta}$ and $\varepsilon := \tilde{\beta}r - 1$ and set $\ell' = a$ and $r' := b$. \square

Remark 3.3. Although of interest in its own right, Proposition 3.1 is in this note only used in the proof of Proposition 2.2. As the proof of Proposition 3.1 is rather lengthy, I'm very grateful to Tom Bridgeland who pointed out the following shortcut to the argument in *Claim 4* in Section 2.

Suppose L' and r' are as in the Lemma and suppose there exists a μ -stable vector bundle E with $\chi(F, E) > 0$. Then $\text{Hom}(F, E) \neq 0$ or $\text{Hom}(E, F) \neq 0$, but the latter is excluded by stability. A non-trivial morphism $F \rightarrow E$ with E μ -stable of slope $\leq \beta$ gives rise to a non-trivial morphism $F[1] \rightarrow E[1]$ in \mathcal{A} . But in the proof of Proposition 2.2 the object $F[1]$ was supposed to be simple and, therefore, any non-trivial morphism from $F[1]$ is injective in \mathcal{A} . However, the quotient G^\bullet (in \mathcal{A} !) of $F[1] \rightarrow E[1]$ satisfies $(c_1(G^\bullet) - \text{rk}(G^\bullet)B.\omega) = -(\ell'.\omega) + r'\beta < 0$, which contradicts $G^\bullet \in \mathcal{A}$.

Finally, one observes that $\chi(F, E) = -\langle v(F), v(E) \rangle$, but if $v(E) = (r + r', \ell + \ell', s')$, then $\langle v(F), v(E) \rangle = (\ell.\ell + \ell') - rs' - (r + r')s < 0$ for $s' \gg 0$. Thus, μ -stable vector bundles E of rank $r + r'$ with $\det(E) \cong L \otimes L'$ and $c_2(E) \gg 0$, which certainly exist, will yield the above contradiction to the minimality of $F[1]$.

Let us now prepare the proof of Proposition 3.1. In the course of the proof we shall make use of the existence of e -stable vector bundles. Here e is a real number, usually positive, and a torsion free sheaf G is called e -stable if for all subsheaves $0 \neq G_1 \subset G$ with $\text{rk}(G_1) < \text{rk}(G)$ one has

$$\mu(G_1) < \mu(G) - \frac{e}{\text{rk}(G_1)}.$$

For $e > 0$ this is in general a stronger version of μ -stability.

O'Grady proved the existence of e -stable vector bundles with large second Chern number: For fixed e , L' , and r' and $c \gg 0$, there exists an e -stable vector bundle G with $\det(G) \cong L'$, $\text{rk}(G) = r'$, and $c_2(G) = c$. The bound can be made effective (see [15] or [8, Thm. 9.11]).

Proof of Proposition 3.1. In the following we let L' and r' be as in Lemma 3.2 and we assume furthermore $r' \geq r$. The vector bundle G will be chosen such that $\det(G) \cong L'$ and $\text{rk}(G) = r'$. The remaining numerical invariant of G is its second Chern number $c_2(G)$, which will have to be chosen large enough.

Consider any extension (possibly trivial)

$$(7) \quad 0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

and a proper saturated subbundle $0 \neq E_1 \subset E$. Then let $F_1 := F \cap E_1$ and $G_1 := E_1/F_1 = \text{Im}(E_1 \longrightarrow G)$.

The following short-hands will be used throughout: $\ell := \deg(L)$, $\ell' := \deg(L')$, $\ell_1 := \deg(F_1)$, $\ell'_1 := \deg(G_1)$, $r_1 := \text{rk}(F_1)$, and $r'_1 := \text{rk}(G_1)$.

i) If $G_1 = 0$, then $F_1 = E_1$. Hence, by the μ -stability of F one finds

$$\mu(E_1) = \mu(F_1) \leq \mu(F) < \mu(E).$$

ii) Suppose $0 < \text{rk}(G_1) < \text{rk}(G)$.

Claim. If G is e -stable for some $e \geq \left(\frac{r'r-r}{r+r'}\right) \left(\frac{\ell'}{r'} - \frac{\ell}{r}\right)$, then $\mu(E_1) < \mu(E)$.

The μ -stability of F yields

$$\begin{aligned} \mu(E_1) &= \frac{\ell_1 + \ell'_1}{r_1 + r'_1} = \frac{\ell_1}{r_1} \cdot \frac{r_1}{(r_1 + r'_1)} + \frac{\ell'_1}{r'_1} \cdot \frac{r'_1}{(r_1 + r'_1)} \\ &\leq \frac{\ell}{r} \cdot \frac{r_1}{(r_1 + r'_1)} + \frac{\ell'_1}{r'_1} \cdot \frac{r'_1}{(r_1 + r'_1)} \end{aligned}$$

(We leave it to the reader to verify that this makes sense also in the case $r_1 = 0$.) Thus, it suffices to show that the right hand side is smaller than $\mu(E) = \frac{\ell + \ell'}{r + r'}$ or, equivalently, that

$$\frac{\ell'_1}{r'_1} < \frac{(\ell + \ell')}{r_1} \cdot \frac{(r_1 + r'_1)}{(r + r')} - \frac{\ell}{r} \cdot \frac{r_1}{r'_1}.$$

Since G is e -stable and $\text{rk}(G_1) < \text{rk}(G)$, one has

$$\frac{\ell'_1}{r'_1} < \frac{\ell'}{r'} - \frac{e}{r'_1}.$$

Hence $\mu(E_1) < \mu(E)$ if

$$\frac{\ell'}{r'} - \frac{e}{r'_1} < \frac{(\ell + \ell')}{r_1} \cdot \frac{(r_1 + r'_1)}{(r + r')} - \frac{\ell}{r} \cdot \frac{r_1}{r'_1}$$

or, equivalently, if

$$\left(\frac{\ell'}{r'} - \frac{\ell}{r}\right) \left(\frac{r'_1 r - r' r_1}{r + r'}\right) < e.$$

The maximum of the left hand side with $0 \leq r_1 \leq r$ and $0 < r'_1 < r'$ is attained for $r_1 = 0$ and $r'_1 = r' - 1$ and yields $\left(\frac{r'r-r}{r+r'}\right) \left(\frac{\ell'}{r'} - \frac{\ell}{r}\right)$.

iii) Consider now the remaining case that $\text{rk}(G_1) = \text{rk}(G)$. Since $E_1 \subset E$ is a saturated and proper subsheaf, F/F_1 is torsion free and $\text{rk}(F_1) < \text{rk}(F)$. If E_1 is μ -destabilizing, then $\mu(E_1) \geq \mu(E)$ or, equivalently,

$$(8) \quad \ell_1 \geq (\ell + \ell') \frac{r_1 + r'}{r + r'} - \ell'_1.$$

Consider

$$\mathcal{S} := \{F_1 \subset F \mid F/F_1 \text{ torsion free, } \mu(F_1) \geq C\}$$

with $C := (\ell + \ell') \frac{r_1 + r'}{r + r'} - \ell'$. Note that C is bigger than the right hand side of (8).

If $\xi \in \text{Ext}^1(G, F)$ denotes the extension class of (7) and $\eta \in \text{Ext}^1(S, F/F_1)$ the extension class of the induced short exact sequence

$$0 \longrightarrow F/F_1 \longrightarrow E/E_1 \longrightarrow S := G/G_1 \longrightarrow 0,$$

then they yield identical classes in $\text{Ext}^1(G, F/F_1)$ under the natural maps $\text{Ext}^1(G, F) \longrightarrow \text{Ext}^1(G, F/F_1)$ respectively $\text{Ext}^1(S, F/F_1) \longrightarrow \text{Ext}^1(G, F/F_1)$.

Suppose we can choose G and $\xi \in \text{Ext}^1(G, F)$ such that:

- (A) For all $F_1 \in \mathcal{S}$ the image of ξ is not contained in the image of $\text{Ext}^1(S, F/F_1) \longrightarrow \text{Ext}^1(G, F/F_1)$ for any torsion quotient $G \twoheadrightarrow S$.

Then E does not admit a destabilizing subsheaf with $\text{rk}(G_1) = \text{rk}(G)$.

iv) To conclude the proof it suffices to show that there exists an e -stable vector bundle G with $\det(G) = L'$, $\text{rk}(G) = r'$, e as in ii), and an extension class $\xi \in \text{Ext}^1(G, F)$ satisfying (A).

First note that due to a lemma of Grothendieck (see [8, Lemma 1.7.9]) the family \mathcal{S} is bounded. Next consider the reflexive hull of F/F_1 which sits in a short exact sequence

$$0 \longrightarrow F/F_1 \longrightarrow (F/F_1)^{\vee\vee} \longrightarrow T \longrightarrow 0$$

for some torsion sheaf T . Applying $\text{Hom}(S, \cdot)$ yields a bijection $\text{Hom}(S, T) \cong \text{Ext}^1(S, F/F_1)$ for any torsion quotient $G \twoheadrightarrow S$. Thus there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}(S, T) & \xrightarrow{\sim} & \text{Ext}^1(S, F/F_1) \\ \downarrow & & \downarrow \\ \text{Hom}(G, T) & \longrightarrow & \text{Ext}^1(G, F/F_1), \end{array}$$

which shows that the image of $\text{Ext}^1(S, F/F_1) \longrightarrow \text{Ext}^1(G, F/F_1)$ is contained in the image of $\text{Hom}(G, T) \longrightarrow \text{Ext}^1(G, F/F_1)$ which is independent of S .

Since \mathcal{S} is a bounded family, the length of the occurring sheaves $T = (F/F_1)^{\vee\vee}/(F/F_1)$ with $F_1 \in \mathcal{S}$ is bounded by say a . Hence, for any $F_1 \in \mathcal{S}$ the dimension of

$$\begin{aligned} & \bigcup_{G \twoheadrightarrow S} \text{Im}(\text{Ext}^1(S, F/F_1) \longrightarrow \text{Ext}^1(G, F/F_1)) \\ & \subset V_{F_1} := \text{Im}(\text{Hom}(G, T) \longrightarrow \text{Ext}^1(G, F/F_1)) \end{aligned}$$

is bounded by $r'a$.

Consider the pre-image $V'_{F_1} \subset \text{Ext}^1(G, F)$ of $V_{F_1} \subset \text{Ext}^1(G, F/F_1)$ under

$$\text{Ext}^1(G, F) \longrightarrow \text{Ext}^1(G, F/F_1).$$

In order to find ξ satisfying (A) it suffices to show that for $c_2(G) \gg 0$ and G generic, the algebraic set $\bigcup_{F_1 \in \mathcal{S}} V'_{F_1} \subset \text{Ext}^1(G, F)$ has dimension strictly smaller than $\dim \text{Ext}^1(G, F)$.

By definition, $\dim(V'_{F_1}) \leq \dim(V_{F_1}) + \text{ext}^1(G, F_1)$. Thus,

$$\dim\left(\bigcup_{F_1 \in \mathcal{S}} V'_{F_1}\right) \leq r'a + \sup\{\text{ext}^1(G, F_1) \mid F_1 \in \mathcal{S}\} + \dim(\mathcal{S}).$$

Write $\text{ext}^1(G, F_1) = \text{hom}(G, F_1) + \text{hom}(F_1, G) - \chi(G, F_1)$. Clearly, if G is μ -stable, then $\text{Hom}(G, F_1) \subset \text{Hom}(G, F) = 0$. Since \mathcal{S} is bounded, there exists a constant μ_0 such that $\mu_0 \leq \mu(\hat{F}_1)$ for all quotients $F_1 \twoheadrightarrow \hat{F}_1$ of a sheaf $F_1 \in \mathcal{S}$.

If G is e -stable with $e \geq \mu(G) - \mu_0 = \frac{\ell'}{r'} - \mu_0$, then $\text{Hom}(F_1, G) = 0$. Indeed, for the image \hat{F}_1 of a non-trivial morphism $F_1 \twoheadrightarrow G$ one would obtain the contradiction $\mu_0 \leq \mu(\hat{F}_1) < \mu(G) - e \leq \mu_0$. (Recall $r' \geq r$.)

The Riemann–Roch formula then shows that for $e \geq \frac{\ell'}{r'} - \mu_0$ and an e -stable vector bundle G the dimension $\text{ext}^1(G, F_1) = -\chi(G, F_1)$ grows like $r_1 c_2(G)$ for $c_2(G) \rightarrow \infty$. On the other hand, $\text{ext}^1(G, F)$ grows at least like $r c_2(G)$. Hence,

$$\dim\left(\bigcup_{F_1 \in \mathcal{S}} V'_{F_1}\right) < \dim \text{Ext}^1(G, F)$$

for $c_2(G) \gg 0$ and G an e -stable vector bundle for any $e \geq \frac{\ell'}{r'} - \mu_0$.

Eventually choose $e \geq \max\left\{\left(\frac{rr'-r}{r+r'}\right)\left(\frac{\ell'}{r'} - \frac{\ell}{r}\right), \frac{\ell'}{r'} - \mu_0\right\}$. \square

Remark 3.4. i) The proof also shows that G (and hence E) can be chosen such that $\det(F)$ and $\det(G)$ are linearly dependent.

ii) The proposition holds true for any torsion free μ -stable sheaf F . Again, G can be chosen locally free, but E would only be torsion free in this more general situation.

iii) One can be a bit more specific about the slope $\mu(E)$. In fact, any slope that could in principle be realized can also be realized as a slope $\mu(E)$. For example if ω and β are both rational, then we can find E such that $\mu(E) = \beta$.

4. FM-PARTNERS VIA μ -STABLE VECTOR BUNDLES

Due to results of Mukai [14] and Orlov [17], one knows that two K3 surfaces X and X' are derived equivalent if and only if X' is a fine moduli space of stable sheaves on X (see also [12, Ch. 10]). A priori ‘stable’ in this context means ‘Gieseker stable’ (and not μ -stable) and the sheaves are just torsion free (and not locally free). However, as will be shown in this section, the stronger result holds true, i.e. one can work with μ -stable locally free sheaves. The result might be known to the experts – the techniques certainly are. In particular, Yoshioka treats this question in [25, Lemma 2.1], but I was not always absolutely sure about the assumptions in [25] and in the article [23] it is based on. In any case, as the explicit statement, crucial for the rest of the paper, does not seem to be in the literature and for the reader’s convenience, we include a complete proof here.

Proposition 4.1. *Two K3 surfaces X and X' are derived equivalent if and only if either $X \cong X'$ or X' is isomorphic to a fine moduli space of μ -stable vector bundles on X .*

Proof. One direction is a special case of Mukai's result. So, we only have to prove that if $D^b(X') \cong D^b(X)$, then either $X \cong X'$ or X' is isomorphic to a fine moduli space of μ -stable vector bundles on X .

i) We shall often need the following fact: If $\Phi : D^b(Y) \xrightarrow{\sim} D^b(Y')$ is an equivalence between two K3 surfaces with $\Phi^H(0, 0, 1) = (0, 0, 1)$, then $Y \cong Y'$. Indeed, Φ^H then induces a Hodge isometry $(0, 0, 1)_Y^\perp \xrightarrow{\sim} (0, 0, 1)_{Y'}^\perp$, and a Hodge isometry of the quotients

$$H^2(Y, \mathbb{Z}) \cong (0, 0, 1)_Y^\perp / (0, 0, 1)\mathbb{Z} \xrightarrow{\sim} (0, 0, 1)_{Y'}^\perp / (0, 0, 1)\mathbb{Z} \cong H^2(Y', \mathbb{Z}).$$

By the Global Torelli theorem this implies $Y \cong Y'$.

ii) Orlov's proof [17] (or [12, Sect. 10.2]) shows that if $D^b(X) \cong D^b(X')$, then X' is isomorphic to a moduli space $M_H(v)$ of Gieseker stable (with respect to H) sheaves on X with Mukai vector $v = (r, a\ell, s)$, where

$$(9) \quad \ell \in \text{NS}(X) \text{ primitive, } \text{g.c.d.}(r, a(\ell.H), s) = 1, \text{ and } a^2(\ell.\ell) = 2rs.$$

Clearly, the last equality is just expressing $\dim(M_H(v)) = \dim(X') = 2$. The fact that r, a, s are coprime not only ensures that every Gieseker semistable sheaf is Gieseker stable, but also that the moduli space is fine (see [8, Cor. 4.6.7]). Moreover, one may assume $r \geq 2$ if X and X' are not already isomorphic.

iii) Observe that if $\text{g.c.d.}(r, a) = 1$ and if H' is a polarization not lying on a wall, then $M_{H'}(v) = M_{H'}(v)^{\mu_{\text{ss}}}$, i.e. every Gieseker H' -stable sheaf with Mukai vector v is μ -stable (see [8, Thm. 4.C.3]). Furthermore, if \mathcal{E} and \mathcal{E}' are the universal families on $X' \times X = M_H(v) \times X$ and $M_{H'}(v) \times X$, respectively, then the equivalence

$$D^b(X') \xrightarrow[\Phi_{\mathcal{E}}]{\sim} D^b(X) \xrightarrow[\Phi_{\mathcal{E}'}^{-1}]{\sim} D^b(M_{H'}(v))$$

sends $(0, 0, 1)$ to $\Phi_{\mathcal{E}'}^{H'^{-1}}(\Phi_{\mathcal{E}}^H(0, 0, 1)) = \Phi_{\mathcal{E}'}^{H'^{-1}}(v) = (0, 0, 1)$. Hence, due to i), one has $X' \cong M_{H'}(v)$, i.e. X' is isomorphic to a fine moduli space of μ -stable sheaves.

iv) Let us now treat the case of Picard number one. The argument used here is not very geometric, as it makes use of a counting of primitive embeddings into the K3 lattice. (Is there a better one?)

According to [7, Thm. 2.1] every K3 surface X' derived equivalent to a given K3 surface X with $\text{Pic}(X) = \mathbb{Z}\ell$ is isomorphic to a moduli space $M_H(v)$ with $v = (r, \ell, s)$ and $\text{g.c.d.}(r, s) = 1$. In particular, the determinant is primitive and this ensures that μ -semistability implies μ -stability (use iii) or, more directly, the assumption $\rho(X) = 1$). The proof of this result relies in an explicit counting of all Fourier–Mukai partners in [16] and a counting

of all Fourier–Mukai partners arising as one of these moduli spaces (see also [19]).

v) For the case of $\rho(X) \geq 2$ we need to modify a given Mukai vector by spherical twists and line bundle twists. This shall be prepared now.

Recall that $\mathcal{O} \in \mathrm{D}^b(X)$ is a spherical object and therefore induces an autoequivalence $T_{\mathcal{O}} : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X)$, the spherical twist. Its action $T_{\mathcal{O}}^H$ on $\tilde{H}(X, \mathbb{Z})$ interchanges the generators of H^0 and H^4 (up to sign) and leaves invariant $H^2(X, \mathbb{Z})$. If v satisfies (9), then $v' := \pm T_{\mathcal{O}}^H(v)$ does as well. (Choose the sign such that the rank of v' is non-negative.) Thus, the moduli space $M_H(v')$ is non-empty and fine. If \mathcal{E}' denotes the universal family on $M_H(v) \times X$, then the cohomological Fourier–Mukai transform induced by the composition

$$\mathrm{D}^b(M_H(v')) \xrightarrow[\Phi_{\mathcal{E}'}]{\sim} \mathrm{D}^b(X) \xrightarrow[T_{\mathcal{O}}^{-1}]{\sim} \mathrm{D}^b(X) \xrightarrow[\Phi_{\mathcal{E}}^{-1}]{\sim} \mathrm{D}^b(M_H(v))$$

maps $(0, 0, 1)$ to $(0, 0, 1)$. Hence, due to i), one has $M_H(v') \cong M_H(v)$. (If $v' = -T_{\mathcal{O}}^H(v)$, then replace $T_{\mathcal{O}}^{-1}$ by $T_{\mathcal{O}}^{-1}[1]$.)

If $\tilde{L} \in \mathrm{Pic}(X)$, then the autoequivalence $\tilde{L} \otimes () : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X)$ maps (on the cohomology) $v \in \tilde{H}(X, \mathbb{Z})$ to $\tilde{v} := \exp(c_1(\tilde{L})) \cdot v$, which once again satisfies (9). As above,

$$\mathrm{D}^b(M_H(\tilde{v})) \xrightarrow[\Phi_{\tilde{\mathcal{E}}}]{} \mathrm{D}^b(X) \xrightarrow[\tilde{L}^* \otimes ()]{\sim} \mathrm{D}^b(X) \xrightarrow[\Phi_{\mathcal{E}}^{-1}]{\sim} \mathrm{D}^b(M_H(v))$$

sends $(0, 0, 1)$ to $(0, 0, 1)$. Hence, $M_H(\tilde{v}) \cong M_H(v)$.

Summarizing, we conclude that the Mukai vector v can be modified at will by $T_{\mathcal{O}}^H$ and $\exp(\tilde{\ell})$ with $\tilde{\ell} \in \mathrm{NS}(X)$ without changing the isomorphism type of the moduli space.

vi) We are now ready to treat the case $\rho(X) \geq 2$. Suppose $v = (r, a\ell, s)$ satisfies (9) and $r + a\ell = \alpha(r' + a'\ell)$ with $\mathrm{g.c.d.}(r, a) = 1$ (or, equivalently, $r' + a'\ell \in (H^0 \oplus H^2)(X, \mathbb{Z})$ is primitive). Since $\rho(X) \geq 2$, we may choose a primitive $\tilde{\ell} \in \mathrm{NS}(X)$ linearly independent of ℓ . Consider

$$\exp(\tilde{\ell}) \cdot v = \left(r, a\ell + r\tilde{\ell}, \tilde{s} := s + r(\tilde{\ell} \cdot \ell)/2 + a(\tilde{\ell} \cdot \ell) \right).$$

Then $r + a\ell + r\tilde{\ell} = \alpha(r' + a'\ell + r'\tilde{\ell})$ and $r' + a'\ell + r'\tilde{\ell}$ primitive. Furthermore, as α divides r and a and $\mathrm{g.c.d.}(r, a, s) = 1$, one has $\mathrm{g.c.d.}(\alpha, \tilde{s}) = 1$.

Hence, for $T_{\mathcal{O}}^H(\exp(\tilde{\ell}) \cdot v)$ rank and determinant are coprime. Thus, either $X \cong X'$ or X' is isomorphic to a moduli space of μ -stable sheaves on X .

vii) In the last step, one has to show that either $X' \cong X$ or X' is isomorphic to a fine moduli space of μ -stable locally free(!) sheaves on X . According to iv) and vi), $X \cong X'$ or X' is isomorphic to a fine moduli space $M_H(v)$ of μ -stable sheaves with $v = (r, \ell, s)$ and $r > 0$. Suppose there exists a μ -stable torsion free sheaf $[E] \in M_H(v)$ which is not locally free.

Consider the natural sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow S \longrightarrow 0,$$

for which S is torsion. Since E is μ -stable and S is concentrated in dimension zero, $E^{\vee\vee}$ is a μ -stable vector bundle. As Mukai observed in [14, Prop. 3.9], the assumption that $[E] \in M_H(v)$ and hence $\mathrm{ext}^1(E, E) = 2$ implies $S \cong k(x)$. Deforming $x \in X$ and the surjection $E^{\vee\vee} \twoheadrightarrow k(x)$ with it, yields a family of kernels of dimension $\dim(X) + \dim(\mathbb{P}(E^\vee(x))) = 2 + r - 1$. Hence, $r = 1$, which implies $X \cong X'$.

Thus all sheaves $[E] \in M_H(v)$ are locally free. \square

5. DERIVED EQUIVALENCE YIELDS ABELIAN EQUIVALENCE

Suppose X and X' are two derived equivalent K3 surfaces. By Proposition 4.1, there exists an isomorphism $X \cong X'$, the trivial case that will not be discussed, or X' is isomorphic to a fine moduli space of μ -stable vector bundles on X . In the latter case we shall denote the universal family on $X \times X'$ by \mathcal{E} . So, for any closed point $y \in X'$ the restriction $\mathcal{E}_y := \mathcal{E}|_{X \times \{y\}}$ is a μ -stable vector bundle on X . Here, μ -stability is taken with respect to a Kähler class $\omega \in \mathrm{NS}(X)_\mathbb{R}$, which can be chosen integral and such that $(\omega, \omega) > 2$. Clearly, the slope $\mu(\mathcal{E}_y)$ is independent of $y \in Y$ and there exists a rational B-field $B \in \mathrm{NS}(X)_\mathbb{Q}$ such that $\mu(\mathcal{E}_y) = (B, \omega) =: \beta$ for all $y \in X'$.

In the following we shall consider the t -structure induced by ω and B (respectively β) and consider its heart

$$\mathcal{A} := \mathcal{A}(\exp(B + i\omega)) = \mathcal{A}(\beta).$$

Remark 5.1. Bridgeland shows that $Z(F^\bullet) = \langle \exp(B + i\omega), v(F^\bullet) \rangle$ is a stability function that satisfies the HN-property (see [4, Sect. 6,9]). So, the t -structure induced by $\exp(B + i\omega)$ and together with Z define a stability condition on $\mathrm{D}^b(X)$.

We shall only use that $Z(F^\bullet) \in \mathbb{R}_{>0} \exp(i\pi\phi(F^\bullet))$ with $0 < \phi(F^\bullet) \leq 1$ (see Remark 1.5) for any $0 \neq F^\bullet \in \mathcal{A}$.

As was proved already by Mukai in [14], the Fourier–Mukai transform

$$\Phi := \Phi_{\mathcal{E}[1]} : \mathrm{D}^b(X') \xrightarrow{\sim} \mathrm{D}^b(X)$$

with kernel $\mathcal{E}[1] \in \mathrm{D}^b(X' \times X)$ is an exact equivalence. Note that for a closed point $y \in X'$ one has

$$\Phi(k(y)) \cong \mathcal{E}_y[1].$$

Furthermore, its inverse $\Psi := \Phi^{-1}$ is the Fourier–Mukai transform $\Phi_{\mathcal{E}^\vee[1]}$ with kernel $\mathcal{E}^\vee[1] \in \mathrm{D}^b(X \times X')$.

The stability condition given by (\mathcal{A}, Z) induces via the equivalence

$$\Psi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X')$$

a stability condition on $D^b(X')$. More precisely, one sets

$$\mathcal{A}' := \Psi(\mathcal{A})$$

and

$$Z'(F^\bullet) := \langle \Psi^H(\exp(B + i\omega)), v(F^\bullet) \rangle = \langle \exp(B + i\omega), v(\Phi^H(F^\bullet)) \rangle.$$

Here, $\Psi^H : \tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X', \mathbb{Z})$ and its inverse $\Phi^H : \tilde{H}(X', \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X, \mathbb{Z})$ are the naturally induced Hodge isometries.

Clearly, $\text{Im}(Z'(F^\bullet)) \geq 0$ for any $F^\bullet \in \mathcal{A}'$ and any $k(y) = \Psi(\mathcal{E}_y[1])$ satisfies $Z'(k(y)) = Z(\mathcal{E}_y[1]) \in \mathbb{R}$. As by Proposition 2.2 the μ -stable vector bundles \mathcal{E}_y yield minimal objects $\mathcal{E}_y[1] \in \mathcal{A}$, this shows that all point sheaves $k(y)$ are minimal objects in \mathcal{A}' , hence stable, of phase $\phi'(k(y)) = 1$.

Next recall that

$$\Psi^H(\exp(B + i\omega)) = \lambda \exp(B' + i\omega')$$

for some ample class $\omega' \in \text{NS}(X')$ and a positive integer λ (see [11, Sect. 5] or [24, Lemma 7.1]). (Note that since Ψ^H is an integral Hodge isometry, λ and ω' are indeed integral.)

We wish to compare the abelian category $\mathcal{A}' \subset D^b(X')$ with the heart $\mathcal{A}(\exp(B' + i\omega'))$ associated to the ample class ω' and the B-field B' .

Proposition 5.2. *The two abelian subcategories \mathcal{A}' and $\mathcal{A}(\exp(B' + i\omega'))$ coincide, i.e.*

$$\mathcal{A}' = \mathcal{A}(\exp(B' + i\omega')) \subset D^b(X').$$

Proof. The proof is an immediate consequence of the discussion in Section 6 of [4]. Bridgeland shows that the heart of any stability condition on $D^b(X')$ for which all point sheaves $k(y)$ are stable of phase one is of the type claimed by the proposition. For the convenience of the reader (and because the result in [4] is not quite phrased as explicitly as our assertion) we include the argument. (Also, we don't really use that \mathcal{A}' is the heart of a stability condition.)

To shorten the notation, we let $\beta' := (B', \omega')$ and denote the abelian category $\mathcal{A}(\exp(B' + i\omega'))$ simply by $\mathcal{A}(\beta')$ (the polarization ω' is understood). Similarly, the torsion pair of $\text{Coh}(X')$ defining $\mathcal{A}(\beta')$ is denoted $(\mathcal{T}(\beta'), \mathcal{F}(\beta'))$.

Bridgeland defines a torsion pair $(\mathcal{T}', \mathcal{F}')$ in $\text{Coh}(X')$ by

$$\mathcal{T}' := \mathcal{A}' \cap \text{Coh}(X') \text{ and } \mathcal{F}' := \mathcal{A}'[-1] \cap \text{Coh}(X'),$$

for which it is easy to verify that its tilt yields \mathcal{A}' . All what is needed to prove this is collected in [4, Lemma 6.1] (see also the arguments in the proof of Proposition 2.2):

- a) Any object $F^\bullet \in \mathcal{A}'$ is concentrated in degree 0 and -1 and $\mathcal{H}^{-1}(F^\bullet)$ is torsion free.
- b) If $F^\bullet \in \mathcal{A}'$ is stable of phase one, then either $F^\bullet \cong k(y)$ or $F^\bullet \cong F[1]$ with F locally free. (See the arguments in the proof of Proposition 2.2.)
- c) $\text{Coh}(X') \subset \mathcal{A}' \cup \mathcal{A}'[-1]$.

Thus, it suffices to prove that $\mathcal{T}' = \mathcal{T}(\beta')$ and $\mathcal{F}' = \mathcal{F}(\beta')$ or, equivalently, $\mathcal{T}(\beta') \subset \mathcal{T}$ and $\mathcal{F}(\beta') \subset \mathcal{F}'$.

One first proves $\mathcal{T}(\beta') \subset \mathcal{T}'$. Let $F \in \mathcal{T}(\beta')$. Due to a) and c) any torsion sheaf is contained in \mathcal{T}' . Since $F/T(F)$ is again in $\mathcal{T}(\beta')$ and \mathcal{T}' is closed under extension, we can assume that F is torsion free. Next consider the HN-filtration of F with quotients F_{i+1}/F_i which are all, due to the definition of $\mathcal{T}(\beta')$, torsion free and μ -semistable with $\mu(F_{i+1}/F_i) > \beta'$. Again using that \mathcal{T}' is closed under extension, we can thus restrict to the case that F is μ -semistable and further, by using the Jordan–Hölder filtration, to F μ -stable. Now consider the decomposition of F with respect to the torsion pair $(\mathcal{T}', \mathcal{F}')$, i.e. the short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with $F_1 \in \mathcal{T}'$ and $F_2 \in \mathcal{F}'$. If both F_1 and F_2 are non-trivial, one obtains the contradiction

$$\beta' \leq \mu(F_1) < \mu(F) < \mu(F_2) \leq \beta'.$$

Hence, either $F \in \mathcal{T}'$ or $F \in \mathcal{F}'$. The latter can be excluded as follows. Any object in \mathcal{F}' is of the form $\Psi(E^\bullet)[-1]$ with $E^\bullet \in \mathcal{A}$. Therefore, $\text{Im}(Z'(F)) = -\text{Im}(Z(E^\bullet)) \leq 0$ contradicting $F \in \mathcal{T}(\beta')$, i.e. $\mu(F) > \beta'$.

Next, one shows $\mathcal{F}(\beta') \subset \mathcal{F}'$. For this purpose pick $F \in \mathcal{F}(\beta')$, which by definition is torsion free and such that all HN-factors have slope $\leq \beta'$. As above, it suffices to show that any μ -stable torsion free sheaf F with $\mu(F) \leq \beta'$ is contained in \mathcal{F}' . The same reasoning as above allows one to conclude that either $F \in \mathcal{F}'$ or $F \in \mathcal{T}'$. If $F \in \mathcal{T}'$, then choose a point $y \in X'$ and a surjection $\varphi : F \twoheadrightarrow k(y)$ in $\text{Coh}(X')$. Considered as a morphism in \mathcal{A}' it is non-trivial and, since $k(y)$ is minimal in \mathcal{A}' , also surjective. Thus, its kernel F_1 is again contained in $\mathcal{T}' \subset \mathcal{A}'$ and satisfies $Z'(F_1) \in \mathbb{R}_{<0}$. Hence $Z'(F) = Z'(k(y)) + Z'(F_1)$. Now continue with F_1 . This leads to $Z'(F) = kZ'(k(y)) + \sum_{i=1}^k Z'(F_i)$ with $Z'(F_i) \in \mathbb{R}_{<0}$ and hence to a contradiction for $k \rightarrow \infty$. \square

By definition Ψ induces an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$. This yields the second part of Theorem 0.1.

Corollary 5.3. *If X and X' are two derived equivalent K3 surfaces, then there exist complexified polarizations $B + i\omega$ and $B' + i\omega'$ on X respectively X' and an equivalence*

$$\mathcal{A}_X(\exp(B + i\omega)) \cong \mathcal{A}_{X'}(\exp(B' + i\omega')).$$

Remark 5.4. Suppose $\Phi : \text{D}^b(X) \xrightarrow{\sim} \text{D}^b(X')$ is any equivalence such that $\Phi^H(\exp(B + i\omega)) = \exp(B' + i\omega')$ up to a positive scalar, e.g. Φ induced by a universal sheaf of Gieseker stable torsion free sheaves. Then $\Phi(\mathcal{A}_X(\exp(B + i\omega)), Z)$ is the heart of a stability condition on X' in the same fibre of $\pi : \text{Stab}(X') \rightarrow \mathcal{P}_0^+(X')$ as $(\mathcal{A}_{X'}(\exp(B + i\omega)), Z')$. If it is contained in Bridgeland's distinguished component $\Sigma(X')$, then there exists an autoequivalence

Ψ of $D^b(X')$ such that $\Psi(\Phi(\mathcal{A}_X(\exp(B + i\omega)))) = \mathcal{A}_{X'}(\exp(B' + i\omega'))$. The above calculation shows that for the Fourier–Mukai equivalence induced by the universal family of μ -stable vector bundles we do not need to check whether the image is contained in $\Sigma(X')$ and can set directly $\Psi = \text{id}$.

6. TWISTOR SPACE INTERPRETATION

Suppose ω is a Kähler class on a not necessarily projective K3 surface X . Identifying ω with its unique Ricci-flat (hyperkähler) representative allows one to write down an explicit model of the associated twistor space $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$. If I is the complex structure defining X and J, K are the complementary ones determined by ω , then \mathcal{X} is $X \times \mathbb{P}^1$ endowed with the complex structure acting by $(\lambda_1 I + \lambda_2 J + \lambda_3 K, I_{\mathbb{P}^1})$ in $(x, \lambda := (\lambda_1 + \lambda_2 + \lambda_3)) \in X \times \mathbb{P}^1$.

In particular, π , which is by definition the second projection, is holomorphic and for any $x \in X$ the curve $L_x := \{x\} \times \mathbb{P}^1 \subset X \times \mathbb{P}^1 = \mathcal{X}$ is a holomorphic section of π , the twistor sections.

- i) To L_x one associates $\mathcal{O}_{L_x} \in \text{Coh}(\mathcal{X})$, which has the property that $\mathcal{O}_{L_x}|_{X=\pi^{-1}(I)} \cong k(x)$.

Suppose E is a holomorphic vector bundle on X which is μ -stable with respect to ω . Then E admits a unique Hermite–Einstein metric. The curvature of the induced Chern connection satisfies the Hermite–Einstein equation $i\Lambda_\omega F_\nabla = \eta \cdot \text{id}$ with $\eta \int_X \omega^2 = 4\pi\mu(E)$. Thus, if $\mu(E) = 0$, the equation reads $i\Lambda_\omega F_\nabla = 0$. The bundle E can be viewed as a complex vector bundle simultaneously on all the fibres $\mathcal{X}_\lambda = \pi^{-1}(\lambda)$ and, as was first observed by Itoh (see e.g. [1, 21]), the $(0, 1)$ -part with respect to λ defines again a $\bar{\partial}$ -operator. Thus, $(E, \nabla^{(0,1)\lambda})$ is a holomorphic vector bundle on \mathcal{X}_λ . Moreover, these bundles glue to a holomorphic vector bundle on \mathcal{X} .

- ii) To any μ -stable vector bundle E on X with $\mu(E) = 0$ one associates a distinguished vector bundle $\mathcal{E} \in \text{Coh}(\mathcal{X})$ with $\mathcal{E}|_X \cong E$.

In the above situation consider a generic fibre \mathcal{X}_λ and $F \in \text{Coh}(\mathcal{X}_\lambda)$. Since \mathcal{X}_λ does not contain any curves, the torsion of F is concentrated in dimension zero and therefore admits a filtration with quotients isomorphic to some $k(x)$.

The fibre \mathcal{X}_λ is endowed with a natural Kähler form $\omega_\lambda = \lambda_1 \omega_I + \lambda_2 \omega_J + \lambda_3 \omega_K$ and \deg and μ are considered with respect to it. Due to the generic choice of λ , one has $\deg(L) = 0$ for any line bundle $L \in \text{Pic}(\mathcal{X}_\lambda)$. In particular, the reflexive hull $F^{\vee\vee}$ is (if not trivial) a μ -polystable vector bundle, which can by the above procedure obtained as a restriction of the direct sum of some \mathcal{E} .

Thus, for generic $\lambda \in \mathbb{P}^1$ the coherent sheaves

$$\mathcal{O}_{L_x}|_{\mathcal{X}_\lambda} \text{ and } \mathcal{E}|_{\mathcal{X}_\lambda}$$

with \mathcal{O}_{L_x} and \mathcal{E} as in i) respectively ii) generate the category $\mathrm{Coh}(\mathcal{X}_\lambda)$, i.e. any coherent sheaf on \mathcal{X}_λ admits a filtration with quotients of this type. This fact is central for the discussion in [22].

Let us now pass from Coh to \mathcal{A} . Although, \mathcal{X}_λ is not projective for generic λ , the category $\mathcal{A}_{\mathcal{X}_\lambda}(\exp(i\omega_\lambda))$ can still be constructed as in Section 1.

Proposition 6.1. *For the generic twistor fibre \mathcal{X}_λ the minimal objects of the abelian category $\mathcal{A}_{\mathcal{X}_\lambda}(\exp(i\omega_\lambda))$ are $k(x)$ with $x \in \mathcal{X}_\lambda$, and $E[1]$, where E is a μ_{ω_λ} -stable vector bundle.*

They generate the abelian category.

Note that the second assertion is neither true for $\mathrm{Coh}(X)$ with X arbitrary (projective or not) nor for $\mathcal{A}(\exp(i\omega))$ in the algebraic case. Indeed, the minimal objects $k(x) \in \mathrm{Coh}(X)$ do not generate locally free sheaves neither do the minimal objects of $\mathcal{A}(\exp(i\omega))$ described by Proposition 2.2 generate $F[1]$ with F locally free of negative slope.

Proof. For the description of the minimal objects of $\mathcal{A} := \mathcal{A}_{\mathcal{X}_\lambda}(\exp(i\omega_\lambda))$ one follows the arguments in Section 2. The proof that the $k(x)$ and $F[1]$ are minimal did not use any projectivity. To show that minimal objects are of this form, observe that all line bundles are of degree zero. Thus, objects of the form $F[1]$ with F locally free and μ -stable of negative slope do not exist.

Again using that line bundles are of degree zero, one proves that \mathcal{X}_λ does not contain any curves. Hence, if $F^\bullet \in \mathcal{A}$, then $\mathcal{H}^0(F^\bullet)$ is a torsion sheaf in dimension zero. For the same reason, $\mathcal{H}^{-1}(F^\bullet)$ is a μ -semistable torsion free sheaf of slope zero. Since such a torsion free sheaf F gives rise to a short exact sequence

$$0 \longrightarrow F^{\vee\vee}/F \longrightarrow F[1] \longrightarrow F^{\vee\vee}[1] \longrightarrow 0$$

in \mathcal{A} , every object in \mathcal{A} does indeed admit a filtration with quotients isomorphic to $k(x)$ or $E[1]$ as claimed. \square

The conclusion of the above discussion is that the minimal objects of $\mathcal{A}_X(\exp(i\omega))$ deform naturally to minimal objects in $\mathcal{A}_{\mathcal{X}_\lambda}(\exp(i\omega_\lambda))$ on the generic twistor fibre, where they generate the abelian category. In this respect, $\mathcal{A}_X(\exp(i\omega))$ behaves better than $\mathrm{Coh}(X)$ itself, as the minimal object, which are simply the $k(x)$'s, do deform but never generate $\mathrm{Coh}(\mathcal{X}_\lambda)$.

Up to now we have only considered $\mathcal{A}(\exp(i\omega))$ or, only virtually more general, $\mathcal{A}(\exp(B+i\omega))$ with $(B.\omega) = 0$. The case $(B.\omega) \neq 0$ can be dealt with in a similar fashion. Roughly, stable vector bundles of slope $\mu = (B.\omega)$ do not deform sideways in the twistor space, but their projectivizations do. This is the point of view in [22], which we will complement by briefly discussing the twistor space associated to the complexified Kähler form $B + i\omega$. This uses the language of Hitchin's generalized Calabi–Yau structures and their period domains. I believe that eventually this will be conceptually the right way of dealing with the general case.

Let us start with a few remarks on the various period domains (see [10] for more details). By definition

$$Q := \{x \mid (x.x) = 0, (x.\bar{x}) > 0\} \subset \mathbb{P}(H^2(X, \mathbb{C})) \cong \mathbb{P}^{21}$$

and

$$\tilde{Q} := \{x \mid \langle x.x \rangle = 0, \langle x.\bar{x} \rangle > 0\} \subset \mathbb{P}(\tilde{H}(X, \mathbb{C})) \cong \mathbb{P}^{23}.$$

Furthermore, we set $Q' := \tilde{Q} \cap \mathbb{P}((H^2 \oplus H^4)(X, \mathbb{C}))$. These three are the period domains of ordinary K3 surfaces, generalized Calabi–Yau structures φ , and generalized Calabi–Yau structures φ with $\varphi_0 = 0$.

For the K3 surface X with its holomorphic volume form σ and with a chosen Kähler class ω one defines

$$\begin{aligned} \tilde{T}(i\omega) &:= \mathbb{P}(\langle \text{Re}(\sigma), \text{Im}(\sigma), \text{Re}(\exp(i\omega)), \text{Im}(\exp(i\omega)) \rangle) \cap \tilde{Q} \\ &\cong \mathbb{P}^3 \cap \tilde{Q}. \end{aligned}$$

The base of the twistor space $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ considered above is via the period map identified with $T(i\omega) := \tilde{T}(i\omega) \cap Q = \tilde{T}(i\omega) \cap Q'$. Note that a general $\mathbb{P}^3 \subset \mathbb{P}^{23}$ would intersect Q in only two points.

For a complexified Kähler form $B + i\omega$ one has the generalized twistor space $\tilde{T}(\exp(i\omega)) = \exp(B) \cdot \tilde{T}(i\omega)$, which for $(B.\omega) \neq 0$ intersect Q only in two points. The restricted twistor space

$$T(\exp(B + i\omega)) = \tilde{T}(\exp(B + i\omega)) \cap Q' = \exp(B) \cdot T(i\omega)$$

parametrizes the generalized Calabi–Yau structures of the form $\sigma_\lambda + \sigma_\lambda \wedge B$, where σ_λ is the holomorphic volume form on \mathcal{X}_λ .

The above discussion should in the case $(B.\omega) \neq 0$ translate into saying that a μ -stable vector bundle F of slope $\mu(F) = (B.\omega)$ naturally deforms to a ‘bundle with respect to $\sigma_\lambda + \sigma_\lambda \wedge B$ ’. A general theory of coherent sheaves for generalized Calabi–Yau structure (of the form $\sigma + \sigma \wedge B$) still awaits to be developed, but for rational B , they should correspond to α_B -twisted sheaves with $\alpha_B = \exp(B^{0,2}) \in H^2(\mathcal{O}^*)$. This fits with the point of view in [22] (see also [9, Prop. 2.3]).

7. STABILITY OF FM-TRANSFORMS

Suppose \mathcal{E} on $X \times M_H(v)$ is a universal family of μ -stable vector bundles on X such that $M_H(v)$ is isomorphic to a K3 surface X' . The Fourier–Mukai transform with kernel \mathcal{E} is an equivalence $\Phi := \Phi_{\mathcal{E}} : \text{D}^b(X) \xrightarrow{\sim} \text{D}^b(X')$. A natural question, studied in a number of papers (see e.g. [1, 21, 26]), is the following:

When is the image $\Phi(E)$ of a μ -stable vector bundle E again a (shifted) μ -stable vector bundle on X' .

(The polarization on X is H and the one on X' has to be chosen appropriately.) It is known (see [26]) that one cannot expect stability in full generality.

The point I wish to make in this section is that the answer is yes for the large class of minimal objects in \mathcal{A} and that our discussion gives a conceptual and straightforward proof for it.

For the following we let X' be a K3 surface isomorphic to a fine moduli space $M_H(v)$ of μ -stable vector bundles on X with Mukai vector $v = (r, \ell, s)$. Denote the universal family on $X \times X'$ by \mathcal{E} and the induced Fourier–Mukai equivalence by $\Phi = \Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(X')$.

Proposition 7.1. *There exists a polarization H' on X' such that for any μ -stable vector bundle E on X with $\mu(E) = -(\ell.H)/r$ one has either*

- $\Phi(E) \cong k(y)[-2]$ if $[E^\vee] \in M_H(v)$ or otherwise
- $\Phi(E) \cong F[-1]$ with F a $\mu_{H'}$ -stable vector bundle on X' .

Proof. As the proof will show, one could more generally consider μ -stability with respect to a Kähler class ω (and not H) on X and then find a Kähler class ω' on X' with the asserted property.

First choose a B-field $B \in \mathrm{NS}(X)_{\mathbb{Q}}$ such that the slope $\mu(\mathcal{E}_y) = (\ell.\omega)/r$ equals $\beta := (B.\omega)$. Then consider the induced stability condition with heart $\mathcal{A}_X(\exp(B + i\omega))$.

Following the discussion in Section 5, $\Phi^H(\exp(B + i\omega)) = \lambda \exp(B' + i\omega')$ for some $\lambda > 0$ and a Kähler class ω' on X' , which is rational if ω was. Furthermore, Proposition 5.2 states that $\Psi := \Phi_{\mathcal{E}^\vee[1]}$ restricts to an equivalence

$$\Psi : \mathcal{A}_X(\exp(B + i\omega)) \xrightarrow{\sim} \mathcal{A}_{X'}(\exp(B' + i\omega')).$$

Clearly, under this equivalence minimal objects are mapped to minimal objects and the minimal objects on either side have been described by Proposition 2.2.

If E is a μ -stable vector bundle on X with $\mu(E) = -\beta$, then $E^\vee[1]$ is a minimal object in $\mathcal{A}_X(\exp(B + i\omega))$. Thus, either $\Psi(E^\vee[1]) \cong k(y)$ for some closed point $y \in X'$ or $\Psi(E^\vee[1]) \cong F[1]$ for some μ -stable vector bundle on X' . As $\Psi^{-1}(k(y)) = \Phi_{\mathcal{E}[1]}(k(y)) \cong \mathcal{E}_y[1]$, the first case occurs precisely if $E^\vee \cong \mathcal{E}_y$ for some $y \in M_H(v)$, i.e. if $[E^\vee] \in M_H(v)$.

To conclude, we observe $\Psi(E^\vee[1]) \cong \Phi_{\mathcal{E}^\vee[1]}(E^\vee[1]) \cong \Phi_{\mathcal{E}^\vee}(E^\vee)[2]$ and by Grothendieck–Verdier duality $\Phi_{\mathcal{E}}(E) \cong \Phi_{\mathcal{E}^\vee}(E^\vee)^\vee[-2] \cong \Psi(E^\vee[1])^\vee$, which is either $k(y)^\vee \cong k(y)[-2]$ or $(F[1])^\vee \cong F^\vee[-1]$. \square

So, roughly μ -stable vector bundles of the same slope (up to sign) as the ones parametrized by the moduli space in question have μ -stable Fourier–Mukai transform.

Remark 7.2. i) In [26] Yoshioka constructs an explicit example of a μ -stable vector bundle with unstable Fourier–Mukai transform. An easy check reveals that due to the numerical conditions he imposes his example is indeed not covered by the proposition.

ii) Bartocci et al. have proved in [1] the result for the case $\mu = 0$. They use the hyperkähler structure of the K3 surface and the interpretation of

μ -stable vector bundles as bundles on the twistor space in a crucial way (see Section 6). In [21] Verbitsky tried to generalize this approach to the case of non-vanishing slope by working with a similar result for the associated projective bundles.

iii) In [24] Yoshioka presents further results on the stability of Fourier–Mukai transforms for the case of Picard number one and $\mu(E^\vee) = \mu(\mathcal{E}_y) + 1/(\mathrm{rk}(E)r)$. This is not covered by our result.

REFERENCES

- [1] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez *A Fourier–Mukai transform for stable bundles on K3 surfaces*. J. Reine Angew. Math. 486 (1997), 1-16.
- [2] A. Bondal, M. van den Bergh *Generators and representability of functors in commutative and non-commutative geometry*. Mosc. Math. J. 3 (2003), 1-36.
- [3] T. Bridgeland *Stability conditions on triangulated categories*. Ann. Math. to appear. math.AG/0212237.
- [4] T. Bridgeland *Stability conditions on K3 surfaces*. math.AG/0307164.
- [5] P. Gabriel *Des catégories abéliennes*. Bull. Soc. Math. France 90 (1962), 323-448.
- [6] D. Happel, I. Reiten, S. Smal *Tilting in abelian categories and quasitilted algebras*. Memoirs of the AMS 575 (1996).
- [7] S. Hosono, B.H. Lian, K. Oguiso, S.-T. Yau *Fourier–Mukai partners of a K3 surface of Picard number one*. In: Vector bundles and representation theory (Columbia, MO, 2002), Contemp. Math. 322 (2002), 43-55.
- [8] D. Huybrechts, M. Lehn *The geometry of moduli spaces of sheaves*. Aspects of Math. E31. Vieweg, Braunschweig, (1997).
- [9] D. Huybrechts, St. Schröer *The Brauer group of analytic K3 surfaces*. IMRN. 50 (2003), 2687-2698.
- [10] D. Huybrechts *Generalized Calabi–Yau structures, K3 surfaces, and B-fields*. Int. J. Math. 16 (2005), 13-36.
- [11] D. Huybrechts, P. Stellari *Equivalences of twisted K3 surfaces*. Math. Ann. 332 (2005), 901-936.
- [12] D. Huybrechts *Fourier–Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs (2006).
- [13] M. Kashiwara, P. Schapira *Sheaves on manifolds*. Grundlehren 292. Springer (1990).
- [14] S. Mukai *On the moduli space of bundles on K3 surfaces, I*. In: Vector Bundles on Algebraic Varieties, Bombay (1984), 341-413.
- [15] K. O’Grady *The irreducible components of the moduli space of vector bundles*. Invent. Math. 112 (1993), 585-613.
- [16] K. Oguiso *K3 surfaces via almost primes*. Math. Res. Lett. 9 (2002), 47-63.
- [17] D. Orlov *On equivalences of derived categories and K3 surfaces*. J. Math. Sci. (New York) 84 (1997), 1361-1381.
- [18] A. Polishchuk *Classification of holomorphic vector bundles on noncommutative two-tori*. Documenta Math. 9 (2004), 163-181.
- [19] P. Stellari *Some remarks about the FM-partners of K3 surfaces with Picard number 1 and 2*. Geom. Dedicata 108 (2004), 1-13.
- [20] M. Verbitsky *Hyperholomorphic bundles over a hyper-Kähler manifold*. J. Alg. Geom. 5 (1996), 633-669.
- [21] M. Verbitsky *Projective bundles over hyperkähler manifolds and stability of Fourier–Mukai transform*. math.AG/0107196.
- [22] M. Verbitsky *Coherent sheaves on general K3 surfaces and tori*. math.AG/0205210.
- [23] K. Yoshioka *Irreducibility of moduli spaces of vector bundles on K3 surfaces*. math.AG/9907001.

- [24] K. Yoshioka *Moduli spaces of stable sheaves on abelian surfaces*. Math. Ann. 321 (2001), 817-884.
- [25] K. Yoshioka *Twisted stability and Fourier–Mukai transform*. math.AG/0106118.
- [26] K. Yoshioka *Stability and the Fourier–Mukai transform*. Math. Z. 245 (2003), 657-665.